Ist Semestral examination
M.Math. Ist year
Subject - Number Theory : Instructor - B.Sury
December 4, 2003

All questions carry equal marks.

Q 1.
Decide whether the following equations have solutions in integers x, y, N:

$$x^2 + 5x - 12 = 31N$$

$$y^2 + 6y + 7 = 317N$$

Q 2.
(a) *Let* $\zeta(s) \sum_{n \geq 1} \frac{a_n}{n^s} = \zeta(s-1)$ *where all these Dirichlet series absolutely converge for Re* $s > t$ *for some finite* t. *Compute* a_p *for any prime* p.
(b) *Let* $\zeta(s) \sum_{n \geq 1} \frac{b_n}{n^s} = 1$ *where both these Dirichlet series absolutely converge for Re* $s > t$ *for some finite* t. *Compute* b_{n^2} *for any* $n > 1$.

OR

Let n *be fixed and* p *be a fixed prime* p *not dividing* n. *If* $f(p)$ *is the order of* p *in the group* $(\mathbf{Z}/n\mathbf{Z})^*$, *and* w *is any* $f(p)$*-th root of unity, prove that there are exactly* $\phi(n)/f(p)$ *characters* χ *of* $(\mathbf{Z}/n\mathbf{Z})^*$ *for which* $\chi(p) = w$. *Hence, conclude that*

$$\prod_{\chi} (1 - \chi(p)T) = (1 - T^{f(p)})^{\phi(n)/f(p)}$$

where χ *runs over all the characters of* $(\mathbf{Z}/n\mathbf{Z})^*$.

Q 3.
Define $\mathbf{Q}_p, \mathbf{Z}_p$ *and prove that each* $a \in \mathbf{Z}_p$ *has a unique series expansion of the form* $\sum_{n \geq 0} a_n p^n$ *where* $0 \leq a_n < p$.

OR

Prove that a unit $a = \sum_{n \geq 0} a_n p^n \in \mathbf{Z}_p^*$ *is a square if, and only if,* a_0 *is a nonzero square modulo* p.

Q 4.
Use the Chevalley-Warning theorem to prove that each element of a finite field is a sum of two squares in it.

OR

Let F be a finite field of cardinality q and let $v : F \to \mathbf{Z}$ be the map defined by $v(0) = q - 1$ and $v(a) = -1$ for $a \neq 0$. If $b \in F$, prove for each m that

$$\sum_{c_1 + \cdots + c_m = b} v(c_1) \cdots v(c_m) = v(b) q^{m-1}.$$

Q 5.

Prove that each z in the upper half-plane can be transformed to a point w in it such that $|Re\ w| \leq 1/2$ and $|w| \geq 1$ by an element of $PSL(2, \mathbf{Z})$.

OR

For each even integer $k \geq 4$, assuming that

$$G_k(z) := \sum_{(m,n) \neq (0,0)} \frac{1}{(mz + n)^k}$$

is absolutely convergent and uniformly convergent on compact subsets of the upper half-plane, prove that $G_k(z)$ is a modular form of weight k which is not a cusp form.

Q 6.

For each natural number n, if $T(n)$ denotes the corresponding Hecke operator and L is a lattice in $\mathbf{C}$, define $T(n)L$. Prove that

$$T(p^n)T(p)L = T(p^{n+1})L + pT(p^{n-1})(pL)$$

for any prime p and any $n \geq 1$.

OR

If $f \in M_{2k}$ is a nonzero cusp form which is a common eigenfunction for all the operators $T(n)$, prove that the coefficient a_1 in the q-expansion $\sum_{n \geq 1} a_n q^n$ of f, is nonzero.